UNIT 1 EXERCISES 11-15

2D GEO

2011A 11. **Answer (B):** Because $AB = 1$, the smallest number of jumps is at least 2. The perpendicular bisector of $\overline{AB}$ is the line with equation $x = \frac{1}{2}$, which has no points with integer coordinates, so 2 jumps are not possible. A sequence of 3 jumps is possible; one such sequence is $(0,0)$ to $(3,4)$ to $(6,0)$ to $(1,0)$.

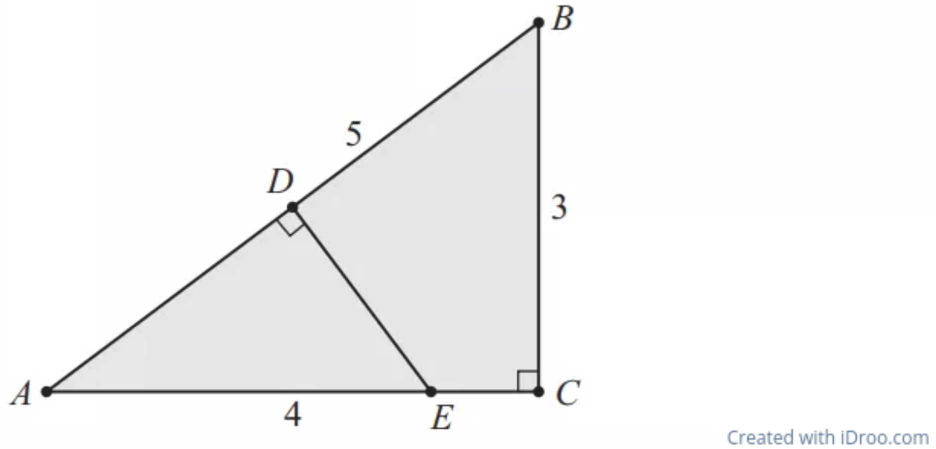
2013A 11. **Answer (C):**

Let $x = DE$ and $y = FG$. Then the perimeter of ADE is $x + x + x = 3x$, the perimeter of $DFGE$ is $x + (y - x) + y + (y - x) = 3y - x$, and the perimeter of $FBCG$ is $y + (1 - y) + 1 + (1 - y) = 3 - y$. Because the perimeters are equal, it follows that $3x = 3y - x = 3 - y$. Solving this system yields $x = \frac{9}{13}$ and $y = \frac{12}{13}$. Thus $DE + FG = x + y = \frac{21}{13}$.

2018A

11. **Answer (D):** The paper's long edge $\overline{AB}$ is the hypotenuse of right triangle ACB, and the crease lies along the perpendicular bisector of $\overline{AB}$. Because $AC > BC$, the crease hits $\overline{AC}$ rather than $\overline{BC}$. Let D be the midpoint of $\overline{AB}$, and let E be the intersection of $\overline{AC}$ and the line through D perpendicular to $\overline{AB}$. Then the crease in the paper is $\overline{DE}$. Because $\triangle ADE \sim \triangle ACB$, it follows that $\frac{DE}{AD} = \frac{CB}{AC} = \frac{3}{4}$. Thus

$$DE = AD \cdot \frac{CB}{AC} = \frac{5}{2} \cdot \frac{3}{4} = \frac{15}{8}.$$



2018B

9. **Answer (E):** Note that the sum of the first 100 positive integers is $\frac{1}{2} \cdot 100 \cdot 101 = 5050$. Then

$$\begin{aligned}\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) &= \sum_{i=1}^{100}\sum_{j=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j \\ &= \sum_{j=1}^{100}\sum_{i=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j \\ &= 100\sum_{i=1}^{100} i + 100\sum_{j=1}^{100} j \\ &= 100(5050+5050) \\ &= 1{,}010{,}000.\end{aligned}$$

OR

Note that the sum of the first 100 positive integers is $\frac{1}{2} \cdot 100 \cdot 101 = 5050$. Then

$$\begin{aligned}\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) &= \sum_{i=1}^{100}\big((i+1)+(i+2)+\cdots+(i+100)\big) \\ &= \sum_{i=1}^{100}(100i+5050) \\ &= 100 \cdot 5050 + 100 \cdot 5050 \\ &= 1{,}010{,}000.\end{aligned}$$

OR

The sum contains 10,000 terms, and the average value of both i and j is $\frac{101}{2}$, so the sum is equal to

$$10{,}000\left(\frac{101}{2}+\frac{101}{2}\right) = 1{,}010{,}000.$$

2006A

12. **(B)** The top of the largest ring is 20 cm above its bottom. That point is 2 cm below the top of the next ring, so it is 17 cm above the bottom of the next ring. The additional distances to the bottoms of the remaining rings are $16\,\text{cm}, 15\,\text{cm}, \ldots, 1\,\text{cm}$. Thus the total distance is

$$20 + (17 + 16 + \cdots + 2 + 1) = 20 + \frac{17 \cdot 18}{2} = 20 + 17 \cdot 9 = 173\,\text{cm}.$$

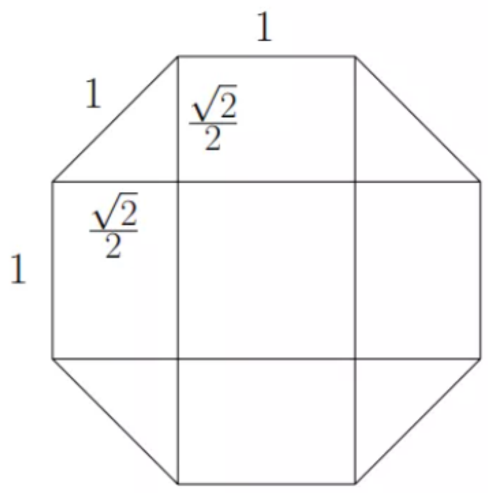
OR

The required distance is the sum of the outside diameters of the 18 rings minus a 2-cm overlap for each of the 17 pairs of consecutive rings. This equals

$$(3+4+5+\cdots+20)-2\cdot 17 = (1+2+3+4+5+\cdots+20)-3-34 = \frac{20 \cdot 21}{2}-37 = 173\,\text{cm}.$$

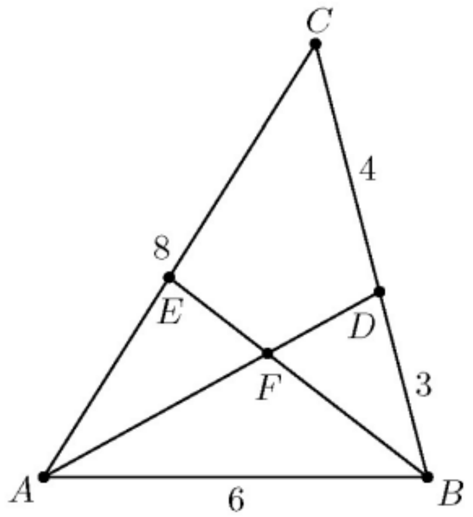
2011B

12. **Answer (A):** Assume the octagon's edge is 1. Then the corner triangles have hypotenuse 1 and thus legs $\frac{\sqrt{2}}{2}$ and area $\frac{1}{4}$ each; the four rectangles are 1 by $\frac{\sqrt{2}}{2}$ and have area $\frac{\sqrt{2}}{2}$ each, and the center square has area 1. The total area is $4 \cdot \frac{1}{4} + 4 \cdot \frac{\sqrt{2}}{2} + 1 = 2 + 2\sqrt{2}$. The probability that the dart hits the center square is $\frac{1}{2+2\sqrt{2}} = \frac{\sqrt{2}-1}{2}$.



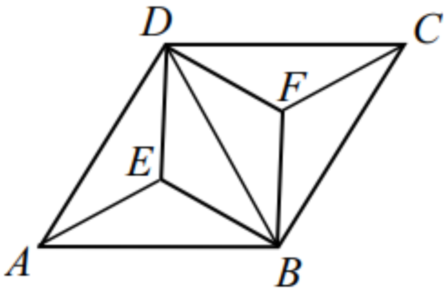
2016A

12. **Answer (C):** Applying the Angle Bisector Theorem to $\triangle BAC$ gives $BD : DC = 6 : 8$, so $BD = \frac{6}{6+8} \cdot 7 = 3$. Then applying the Angle Bisector Theorem to $\triangle ABD$ gives $AF : FD = 6 : 3 = 2 : 1$.



Note: More generally the ratio $AF : FD$ is $(AB + CA) : BC$, which equals $2 : 1$ whenever AB, BC, CA forms an arithmetic progression.

2006B 13. **(C)** Since $\angle BAD = 60^\circ$, isosceles $\triangle BAD$ is also equilateral. As a consequence, $\triangle AEB$, $\triangle AED$, $\triangle BED$, $\triangle BFD$, $\triangle BFC$, and $\triangle CFD$ are congruent. These six triangles have equal areas and their union forms rhombus $ABCD$, so each has area $24/6 = 4$. Rhombus $BFDE$ is the union of $\triangle BED$ and $\triangle BFD$, so its area is 8.

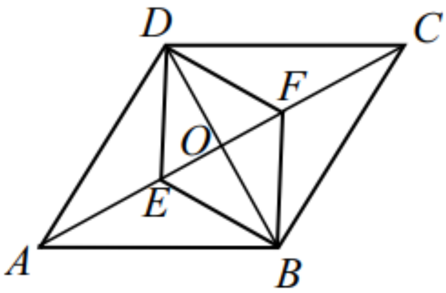


OR

Let the diagonals of rhombus $ABCD$ intersect at O. Since the diagonals of a rhombus intersect at right angles, $\triangle ABO$ is a $30-60-90^\circ$ triangle. Therefore $AO = \sqrt{3} \cdot BO$. Because AO and BO are half the length of the longer diagonals of rhombi $ABCD$ and $BFDE$, respectively, it follows that

$$\frac{\text{Area}(BFDE)}{\text{Area}(ABCD)} = \left(\frac{BO}{AO}\right)^2 = \frac{1}{3}.$$

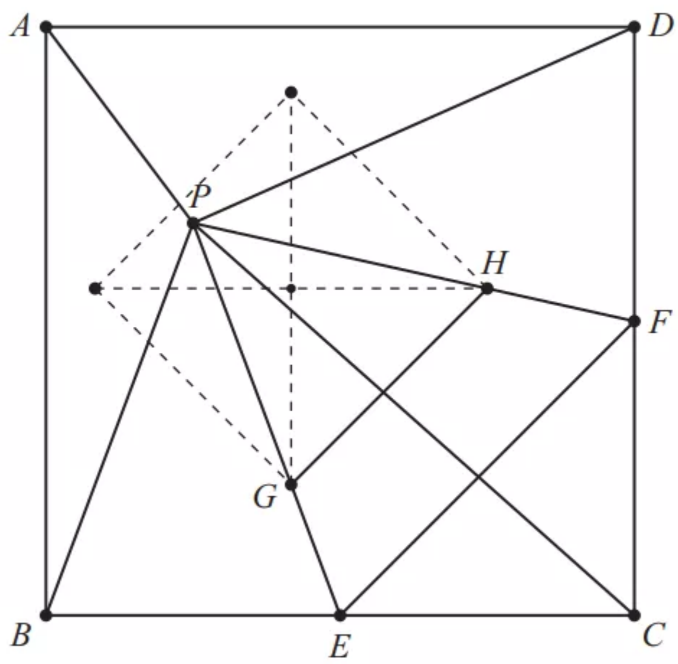
Thus the area of rhombus $BFDE$ is $(1/3)(24) = 8$.



2006A 13. **(E)** Let r, s, and t be the radii of the circles centered at A, B, and C, respectively. Then $r + s = 3$, $r + t = 4$, and $s + t = 5$, from which $r = 1$, $s = 2$, and $t = 3$. Thus the sum of the areas of the circles is

$$\pi(1^2 + 2^2 + 3^2) = 14\pi.$$

2018B 13. **Answer (C):** Let E and F be the midpoints of sides $\overline{BC}$ and $\overline{CD}$, respectively. Let G and H be the centroids of $\triangle BCP$ and $\triangle CDP$, respectively. Then G is on $\overline{PE}$, a median of $\triangle BCP$, a distance $\frac{2}{3}$ of the way from P to E. Similarly, H is on $\overline{PF}$ a distance $\frac{2}{3}$ of the way from P to F. Thus $\overline{GH}$ is parallel to $\overline{EF}$ and $\frac{2}{3}$ the length of $\overline{EF}$. Because $BC = 30$, it follows that $EC = 15$, $EF = 15\sqrt{2}$, and $GH = 10\sqrt{2}$. The midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ form a square, so the centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ also form a square, and that square has side length $10\sqrt{2}$. The requested area is $\left(10\sqrt{2}\right)^2 = 200$.



OR

Place the figure in the coordinate plane with $A = (0, 30)$, $B = (0, 0)$, $C = (30, 0)$, $D = (30, 30)$, and $P = (3x, 3y)$. Then the coordinates of the centroids of the four triangles are found by averaging the coordinates of the vertices: $(x, y + 10)$, $(x + 10, y)$, $(x + 20, y + 10)$, and $(x + 10, y + 20)$. It can be seen that the quadrilateral formed by the centroids is a square with center $(x + 10, y + 10)$ and vertices aligned vertically and horizontally. Its area is half the product of the lengths of its diagonals, $\frac{1}{2} \cdot 20 \cdot 20 = 200$.

Note: As the solutions demonstrate, the inner quadrilateral is always a square, and its size is independent of the location of point P. The location of the square within square $ABCD$ does depend on the location of P.